# 2015 MBA Bootcamp

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## Contents

# Introduction

A Master of Business Administration (MBA) is a graduate degree designed to develop the skills required for careers in management. Among those skills is decision-making. Decision making is often done methodically. A method usually implies the application of logical and mathematical reasoning. Therefore, having a good understanding of mathematics and logic is a valuable asset for any MBA student.

Decision-making often rely on the use of mathematical relationships to represent real-world relationships.

A basic knowledge of algebra will be applied in many courses. The basic concepts of calculus (differential and integral) provide important conceptual foundations for economics, finance, management science and statistics. You will need to understand the basic notions of optimization, rates of change, and areas under curves; for many managerial applications (e.g., marginal costs (rate of change of total cost), total revenue or probability (which are represented by areas under curves), and maximizing benefit or minimizing loss subject to operating constraints (applying optimization principles).

# Order of operations

| Operation | Syntax |
|---|---|
| raising $X$ to power $n$ | $X^n$ |
| taking $n^{th}$ root of $X$ | $\sqrt[n]{X}$ or $X^{\frac{1}{n}}$ |

| Operation | Syntax |
|---|---|
| multiplication | $\times$ |
| division | $\div$ |
| addition | $+$ |
| subtraction | $-$ |

a. An MBA student should know the order of operations as represented in the above chart of the six basic operations.

b. Do the operations in the top two rows first (i.e., roots and powers).

c. Do the operations in third and fourth rows next (i.e., division and multiplication).

d. Finally, do the operations in the last two rows (i.e., addition and subtraction).

e. 
- If you run across two power/root operation work them in the order in which you encounter them, from left to right.
- If you run across two multiplication/division operation work them in the order in which you encounter them, from left to right.
- If you run across two addition/subtraction operation work them in the order in which you encounter them, from left to right.

f. Alternatively, you may recall the mnemonic you most likely first heard in junior high for remembering the order of operations: **"Please Excuse My Dear Aunt Sally"** where:

- "P" tells you to work within parentheses first,
- "E" reminds you to do exponents next,
- "M" and
- "D" say to do multiplication and division next, and
- "A" and "S" say addition and subtraction come last.

Sample problems:

i. Simplify: $6 - 5[4 - 3^2(2 - X)]$

- answer: $76 - 45X$

ii. Sample problem illustrating that parentheses affect the order of operations:

- Say $X = 4$
- $6 - 5[4 - 3^2(2 - X)] = -104$
- $6 - 5[4 - 3^2 \times 2 - X] = 96$
- $6 - 5 \times 4 - 3^2 \times 2 - X = -36$

```
X = 4
6-5*(4-3^2*(2-X))
```

```
[1] -104
```

```
6-5*(4-3^2*2-X)
```

```
[1] 96
```

```
6-5*4-3^2*2-X
```

```
[1] -36
```

iii. The present value of an ordinary annuity formula is: $PVA = PMT\left(\frac{1-\frac{1}{(1+i)^n}}{i}\right)$.

Suppose you win a lottery that pays \$1,000 per year for 20 years and the yearly interest rate is 10%. Using the "order of operations" rules, can determine the present value of your winnings?

*Step by step evaluation:*

```
1000*((1- (1/(1.10)^20))/0.10)
1000*((1- (1/(1.10)^20))/0.10)
1000*((1- (1/6.7275))/0.10)
1000*((1- 0.1486436)/0.10)
1000*(0.8513564/0.10)
1000*8.513564
```

```
[1] 8513.564
```

*Using pre-programmed functions (R)*

```
library(tvm)
npv(i=.10,cf=rep(1000,20),ts=1:20)
```

```
[1] 8513.564
```

# Common rules of algebra

$$a + b = b + a$$

$$1 \times a = a$$

$$(a + b) + c = a + (b + c)$$

$$aa^{-1} = 1 \text{ for } a \neq 0$$

$$a + 0 = a$$

$$(-a)(b) = a(-b) = -ab$$

$$a + (-0) = a$$

$$(-a)(-b) = ab$$

$$(ab)c = a(bc)$$

$$a(b + c) = ab + ac$$

$$ab = ba$$

$$(a + b)c = ac + bc$$

## Examples:

- In the expression, $(-2WX^2Y^4)^3$, the $3^{rd}$ power can be distributed over the factors that are being multiplied inside the parentheses, giving: $(-2)^3(W)^3(X^2)^3(Y^4)^3$. In simplest form, the result is: $-8W^3X^6Y^{12}$

- In $(X + Y)^2$, the $2^{nd}$ power cannot be distributed over the terms that are being added inside the parentheses. In other words, $(X + Y)^2 \neq X^2 + Y^2$. The terms in parentheses can be squared as follows:

$$(X + Y)^2 = (X + Y)(X + Y)$$

The solution, $X^2 + 2XY + Y^2$, is found by multiplying the first term in the first set of parentheses by the first term and then the second term in the second set of parentheses, and then the second term in the first set of parentheses is multiplied times the first term and the second term in the second set of parentheses as follows.

$$
\begin{aligned}
(X + Y)^2 &= (X + Y)(X + Y) \\
&= X \times X + X \times Y + X \times Y + Y \times Y \\
&= X^2 + 2XY + Y^2
\end{aligned}
$$

- A negative fraction (i.e., minus two-thirds) can be written three different ways, $-\frac{2}{3}$, $\frac{-2}{3}$ or $\frac{2}{-3}$.

## Rules of exponents

- $a^n \times a^m = a^{n+m}$
- $(a^n)^m = a^{nm}$
- $a^{-n} = \frac{1}{a^n}$
- $(ab)^n = a^n \times b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^{\frac{n}{m}} = \sqrt[m]{a^n}$
- $\frac{a^n}{a^m} = a^{n-m}$

Example Problem: $\sqrt[5]{32^3} = 32^{3/5} = 8$. Note: The $5^{th}$ root of 32 is 2 and 2 raised to the $3^{rd}$ power is 8.

## Difference between an expression and an equation

An expression is a collection of letters, numbers, and operations. Some expressions can be simplified and some cannot. An expression can be multiplied only by the number 1, because multiplying by the number 1 doesn't change the value of the expression.

- You are asked to simplify a complicated algebraic expression. After much work you correctly simplify it to $\frac{x^3}{5} + 2X^2 + 9X$. Then in an effort to further simplify, you multiply the expression by 5 to eliminate fractions so that you have : $X^3 + 10X^2 + 45X$. Are you correct in doing this?

6

- Answer: No. It is incorrect to multiply an expression by any number other than 1.

- Using the letters "X" and "Y", write an expression that represents "the square root of the sum of the squares of the two quantities."

  - Answer: $\sqrt{X^2 + Y^2}$

- You are asked to simplify a complicated algebraic expression. After much work you correctly simplify it to $3X - 6$. You then express your final answer as $X = 2$. Are you correct in doing this?

  - Answer: No. It is incorrect to set the expression equal to zero and solve. An expression can't be solved. Only an equation or an inequality can be solved. The correct simplification of the expression stops with $3X - 6$.

In contrast, an equation consists of two expressions with an equal sign between them. An equation is a complete sentence, with the equal sign serving as the verb, that makes a claim—the expression on the left side of the equal sign has the same value as the expression on the right side of the equal sign. You can do things to an equation that you can't do to an expression. For example, you can multiply both sides of an equation by 5, or 1, or divide both sides of the equation by 2 or 3. You may also be able to solve an equation, which means finding all values of the variables that, when individually substituted back into the equation, produce a true statement.

- After working through a complicated algebraic equation, you get to a point where you have the following: $\frac{1}{5}Y = \frac{X^3}{5} + 2X^2 + 9X$. Next you decide to multiply both sides of the equation by 5 to eliminate fractions. Can you do this?

  - Answer: Yes. You can multiply both sides of an equation by the same number. The result, in this case, is: $Y = X^3 + 10X^2 + 45X$. Additionally, you can divide both sides of an equation by the same number, add or subtract a constant to both sides of the equation, take the square root of both sides of an equation, etc.

## Proper notation

In mathematics, everything you write has to be written correctly; there is no room for ambiguity. You must use grouping symbols (parentheses, brackets, braces, etc.) whenever failing to use them would change the meaning of an expression or equation. It's better to err on the side of using too many grouping symbols rather than using too few.

- Example: Suppose you are trying to solve the equation $X + 3 = \frac{4}{X+5}$. To clear fractions, we would begin by multiplying both sides of the equation by X + 5. But, in multiplying the left side of the equation by $X + 5$, you cannot write: $X + 3 \times X + 5$. That result would be $X + 3X + 5 = 4X + 5$. You must multiply both sides of the equation by $X + 5$ as follows:

$$\begin{aligned}
X + 3 &= \frac{4}{X+5} \\
(X+5)(X+3) &= \frac{4}{X+5}(X+5) \\
X^2 + 8X + 15 &= 4 \\
X^2 + 8X + 15 - 4 &= 4 - 4 \\
X^2 + 8X + 11 &= 0
\end{aligned}$$

If you want to show that a negative number is being squared, you must use grouping symbols.

- For example, in $(-6)^2$, the squaring applies to the minus sign and the 6 so that you have $(-6)(-6) = 36$. If you write $-6^2$, the squaring applies only to the 6 but not to the minus sign so that the result would be $-36$.

- Simplify $48X^2Y^0$ .

  - Answer: The result is $48X^2$ since the 0 exponent applies only to $Y$ and $Y^0$ is 1.

- Simplify $(48X^2Y)^0$ .

  - Answer: The result is 1 since the 0 exponent applies ot everything in the parentheses.

## Summation

Let $X$ be cost in US dollars. If you are given n=3 values of X: $X_1 = 10$, $X_2 = 30$ and $X_3 = 5$, then:

- The following expression means sum the values of X, starting at $X_1$ and ending with $X_n$.

$$\sum_{i=1}^{n} X_i$$

- In this particular case we expand the sum as:

$$\sum_{i=1}^{n=3} X_i = X_1 + X_2 + X_3 = 10 + 30 + 5$$

# Zero is a special number

$$\frac{0}{\text{non-zero}} = 0$$

$$\frac{\text{non-zero}}{0} \text{ is undefined}$$

$$\frac{0}{0} \text{ is undefined}$$

We take as a definition:

- any value (except 0) raised to the 0 power is 1.

- $0^0$ is undefined because it is equivalent to $\frac{0}{0}$.

# Solving equations

Some equations can be solved easily. Consider, for example, the equation $3X + 10 = 28$. This is a simplistic equation that could be solved by asking ourselves, "What must be added to 10 to get 28?"

- The answer is 18. So, $3X$ must be 18. Because 3 times X is 18, X must be 6. There are no other solutions.

In more complicated cases, we need a more systematic procedure for solving equations. Two equations that have exactly the same solution are called equivalent. The main principle to remember is the *Golden Rule* of mathematics — *whatever thou doest unto the left side of the equation, thou must also do unto the right side of the equation.* These action include, but are not limited to, the following:

- add the same number to both sides of the equation.
- subtract the same number from both sides of the equation.
- multiply both sides of the equation by the same number $\neq 0$.
- divide both sides of the equation by the same number $\neq 0$.

**Examples**

$$
\begin{aligned}
3X + 10 &= X + 4 \\
3X + 10 - 10 &= X + 4 - 10 \text{ [subtract 10 from both sides of equation]} \\
3X &= X - 6 \\
3X - X &= X - X - 6 \text{ [subtract X from both sides of equation]} \\
2X &= -6 \\
\frac{2X}{2} &= \frac{-6}{2} \text{ [divide both sides of equation by 2]} \\
X &= -3
\end{aligned}
$$

Note that the equation in the first paragraph of this section could have been solved using the same principles.

$$
\begin{aligned}
3X + 10 &= 28 \\
3X + 10 - 10 &= 28 - 10 \text{ [subtract 10 from both sides of equation]} \\
3X &= 18 \\
\frac{3X}{3} &= \frac{18}{3} \text{ [divide both sides of equation by 3]} \\
X &= 6
\end{aligned}
$$

Quadratic equations are even more complex. Suppose we encounter the equation $12X^2 - 16X - 3 = 0$. We could try to find the values of X that satisfy the equation by trial and error. However, it would not be easy to find the solutions which are $X = \frac{3}{2}$ and $X = -\frac{1}{6}$. Assuming the equation cannot be factored (more about that later), or at least we don't recognize the factors, then we can substitute into the general quadratic formula:$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where "a" is the coefficient of the squared term, "b" is the coefficient of the linear term, and "c" is the constant to determine the two values of X (i.e., $X = \frac{3}{2}$ and $X = -\frac{1}{6}$).

# Solving Systems of Equations

Sometimes we have two equations with two unknowns (or three equations with three unknowns, and so on). These types of equations can be solved simultaneously by substitution of by using the *Gauss-Jordan reduction method*. Suppose we have the following two equations in two unknowns.

$$
\begin{aligned}
2X + 3Y &= 18 \\
3X - 4Y &= -7
\end{aligned}
$$

## Substitution Procedure

We could solve the first equation for X by subtracting $3Y$ from both sides of the equation and then dividing both sides of the equation by 2 as follows.

$$
\begin{aligned}
2X + 3Y &= 18 \\
2X + 3Y - 3Y &= 18 - 3Y \\
2X &= 18 - 3Y \\
\frac{2X}{2} &= \frac{18}{2} - \frac{3Y}{2} \\
X &= 9 - \frac{3}{2}Y
\end{aligned}
$$

Next we can substitute this value of X for $X$ in the second equation.

$$
\begin{aligned}
3X - 4Y &= -7 \\
3[9 - \frac{3}{2}Y]\check{\ }4Y &= -7 \\
27 - \frac{9}{2}Y - 4Y &= -7 \\
2[27 - \frac{9}{2}Y - 4Y] &= 2(-7) \\
54 - 9Y - 8Y &= -14 \\
54 - 17Y &= -14 \\
54 - 54 - 17Y &= -14 - 54 \\
-17Y &= -68 \\
\frac{-17Y}{-17} &= \frac{-68}{-17} \\
Y &= 4
\end{aligned}
$$

Now we can substitute $Y = 4$ into either of the equations to solve for X.

$$
\begin{aligned}
2X + 3Y &= 18 \\
2X + 3(4) &= 18 \\
2X + 12 &= 18 \\
2X + 12 - 12 &= 18 - 12 \\
2X &= 6 \\
\frac{2X}{2} &= \frac{6}{2} \\
X &= 3
\end{aligned}
$$

OR

$$\begin{aligned} 3X - 4Y &= -7 \\ 3X - 4(4) &= -7 \\ 3X - 16 &= -7 \\ 3X - 16 + 16 &= -7 + 16 \\ 3X &= 9 \\ \frac{3X}{3} &= \frac{9}{3} \\ X &= 3 \end{aligned}$$

## Examples:

- You should be able to solve a linear (first-degree) equation or an equation that can be turned into a linear equation.

Example: $\frac{-5}{2X+3} = \frac{6}{7X-1}$

- Answer: **Cross multiply**

$$\begin{aligned} \frac{-5}{2X+3} &= \frac{6}{7X-1} \\ -5(7X-1) &= 6(2X+3) \text{ [**Cross multiply **]} \\ -35X + 5 &= 12X + 18 \\ -35X + 5 - 5 &= 12X + 18 - 5 \\ -35X &= 12X + 13 \\ -35X - 12X &= 12X - 12X + 13 \\ -47X &= 13 \\ \frac{-47X}{-47X} &= \frac{13}{-47} \\ X &= -\frac{13}{47} \end{aligned}$$

Example: Solve for X in the equation: $aX + 3Y = X + 7$

- Answer: First, get all the X terms on the left side of the equation and all the other terms on the right side of the equation.

$$\begin{aligned}
aX + 3Y &= X + 7 \\
aX - X + 3Y &= X - X + 7 \\
aX - X + 3Y - 3Y &= 7 - 3Y \\
aX - X &= 7 - 3Y \\
X(a-1) &= 7 - 3Y \\
\frac{X(a-1)}{(a-1)} &= \frac{(7-3Y)}{(a-1)} \\
X &= \frac{(7-3Y)}{(a-1)}
\end{aligned}$$

Example: Solve the equation $5X^2 - 3X = 4$

- Answer:

$$\begin{aligned}
5X^2 - 3X - 4 &= 4 - 4 \\
5X^2 - 3X - 4 &= 0
\end{aligned}$$

This equation cannot be factored; therefore, substitute coefficients into general quadratic formula where coefficient of squared term is "a", coefficient of linear term is "b", and constant is "c".

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(5)(-4)}}{2(5)} \\
x &= \frac{3 \pm \sqrt{9 + 80}}{10} \\
x &= \frac{3 \pm \sqrt{89}}{10}
\end{aligned}$$

## Gauss-Jordan Reduction Method

Alternatively, we could have created linear combinations of the two equations to eliminate one of the variables. This procedure is usually best if the solutions may involve fractions or decimals. Let's identify the two equations as equation I and equation II as follows.

$$[\text{I}]\ 2X + 3Y\ =\ 18$$
$$[\text{II}]\ 3X - 4Y\ =\ -7$$

If we multiply the first equation by 4 and the second equation by 3 then we will have $12Y$ in the first equation and $-12Y$ in the second equation. By adding the two equations the Y's would add out.

$$4(\text{I})\ \rightarrow 8X + 12Y\ =\ 72$$
$$+\ 3(\text{II})\ \rightarrow 9X - 12Y\ =\ -21$$
$$\overline{\phantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}$$
$$17X\ =\ 51$$
$$17\frac{X}{17}\ =\ \frac{51}{17}$$
$$X\ =\ 3$$

Now, substitute $X = 3$ into either of the equations to obtain the value of Y.

$$2X + 3Y\ =\ 18$$
$$2(3) + 3Y\ =\ 18$$
$$6 + 3Y\ =\ 18$$
$$6 - 6 + 3Y\ =\ 18 - 6$$
$$3Y\ =\ 12$$
$$\frac{3Y}{3}\ =\ \frac{12}{3}$$
$$Y\ =\ 4$$

OR

$$3X - 4Y\ =\ -7$$
$$3(3) - 4Y\ =\ -7$$
$$9 - 4Y\ =\ -7$$
$$9 - 9 - 4Y\ =\ -7 - 9$$
$$-4Y\ =\ -16$$
$$\frac{-4Y}{-4}\ =\ \frac{-16}{-4}$$
$$Y\ =\ 4$$

# The equation of a straight line

The general form of the equation of a straight line is $Y = a + bX$. When a straight line is graphed on a Cartesian coordinate system, the dependent variable, Y, is ordinarily graphed on the vertical axis and the independent variable, X, is ordinarily graphed on the horizontal axis. The one major exception to this rule is demand and supply equations where the independent variable, price, appears on the vertical axis and the dependent variable, quantity, appears on the horizontal axis.

In the general form of a straight line, $Y = a + bX$, is frequently referred Y to as the slope-intercept form of a straight line because the constant, a, is the vertical axis intercept of the straight line and the coefficient, b, is the slope of the line where slope is defined as: $Slope = \frac{(\text{vert}\Delta)}{(\text{horz}\Delta)}$. You may also see slope described as "rise divided by run" or by "$\frac{\Delta Y}{\Delta X}$." So, in the equation, $Y = 5 - \frac{1}{4}X$, the vertical axis intercept is 5 and the $\frac{1}{4}$ equation has a slope of which means for every 1 unit of vertical change 1 there are 4 units of horizontal change. The line is graphed to the right.
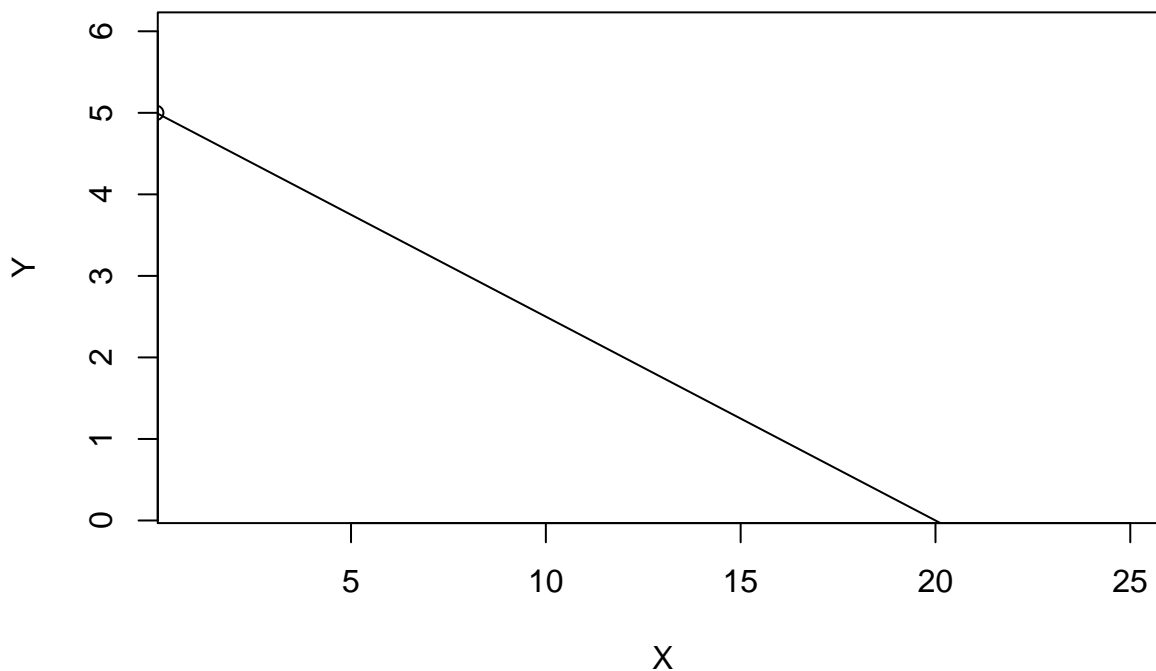


Given a linear equation, the vertical axis intercept can be found by substituting 0 for X in the equation since at $X = 0$ is where the line crosses the vertical axis. The horizontal axis intercept can be found by substituting 0 for Y in the equation and solving for X since at $Y = 0$ is where the line crosses the horizontal axis.

The convention in economics of placing price, the independent variable, on the vertical axis and quantity, the dependent variable on the horizontal axis is generally credited to Alfred Marshall who began this practice in 1890 in *Principles of Economics*, arguably the most popular book of the times for teaching economics. Of particular note to today's students who are familiar with seeing demand and supply diagrams in nearly every page of the textbook, the demand and supply diagrams appeared only in footnotes in this forerunner to today's

principles of economics textbooks.

Since the variables are on the wrong axes (i.e., the independent variable is on the vertical axis and the dependent variable is on the horizontal axis), whenever the general form of a straight line equation is used to write the equation of a demand curve, it is referred to as the "inverse" form of the demand curve. To put the demand curve in "standard" form (i.e., a form where the dependent variable is to the left of the equal sign, the inverse form of the demand equation must be solved for "Q". Note in the diagram to the right, another convention in economics is to label the axes with letters that identify the variables. So, rather than "X" and "Y", "P" is used for Price on the vertical axis and "Q" is used for Quantity on the horizontal axis. In this case, using the general form of a straight line, $P = a + bQ$, to determine the "inverse" form of the demand curve, we have $P = 5 - \frac{1}{8}Q$. To determine the "standard" form of the demand curve, we must solve for Q as follows.
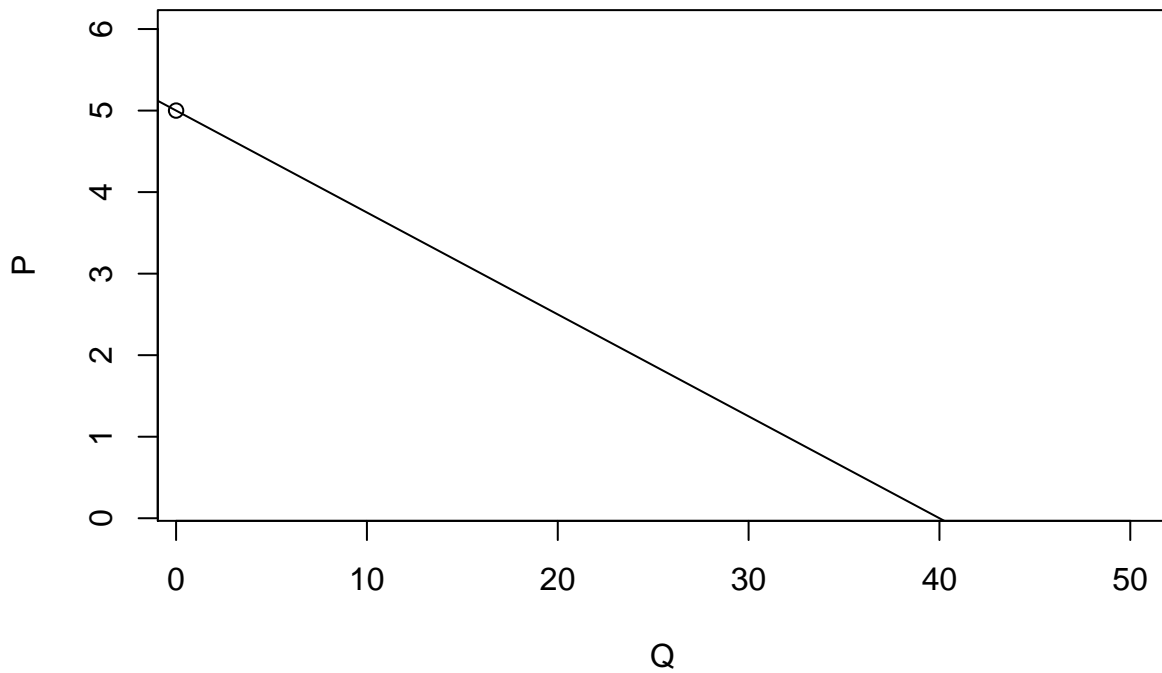
$$P = 5 - \frac{1}{8}Q \text{ ["inverse" form of the demand equation]}$$

$$P + \frac{1}{8}Q = 5 - \frac{1}{8}Q + \frac{1}{8}Q$$

$$P + \frac{1}{8}Q = 5 \text{ [subtract P from both sides of the equation]}$$

$$\frac{1}{8}Q + P - P = 5 - P$$

$$\frac{1}{8}Q = 5 - P$$

$$8\frac{1}{8}Q = 8(5 - P) \text{ [multiply both sides of the equation by 8]}$$

$$Q = 40 - 8P \text{ ["standard" form of the demand equation]}$$

Notice that we began with the "inverse" form of the demand equation in which the independent variable, P, was to the left of the equal sign and we ended with the "standard" form of the demand equation with the dependent variable, Q, on the left side of the equal sign.

## Examples

- The linear equation $Y = 0.22X + 10,000$ gives the relationship between a person's taxable income, X, in dollars, and the amount of federal income tax, Y, that the person owes, also in dollars. Explain, in this particular situation, what the slope means.

  - Answer: The slope of the line tells us that for each additional dollar in taxable income that a person will owe an additional $0.22 in taxes.

- Without using a calculator, simplify the following:

1. $0.9 - 0.23 + 2.7 - 8$
2. $2\frac{2}{3} - 4\frac{3}{4}$
3. $\frac{\frac{2}{3}}{\frac{4}{4}}$
4. $\frac{\frac{12}{6}}{5}$
5. 6 is 12% of what number?
6. My $800 stock portfolio increased by 20% and then decreased by 20%. How much was my stock portfolio worth after these two changes?
7. Simplify: $X^4 - X^3$
8. Simplify: $\frac{X^{a+b}}{X^{a-b}}$
9. Simplify: $5^{-2} - 5^{-1}$

- Answers:

1. -4.63
2. $-\frac{25}{12}$
3. $\frac{1}{6}$
4. 10
5. 50
6. \$768
7. Although you might be tempted to subtract exponents, the rule that involves subtraction of exponents requires that the two terms with the same base are being divided. That's not the case here, so you can't use the rule. The expression, $X^4 - X^3$, cannot be combined or simplified. It is in simplest form. It can be factored into $X^3(X-1)$, but that really isn't simplifying.
8. $X^{2b}$. This time we have a common base and are dividing, so the rule that involves subtracting exponents applies.
9. $-\frac{4}{25}$

# Dealing with fractions

Canceling (reducing) in a fraction is allowable if both the numerator and denominator are in factored form. So-called "partial canceling" is incorrect.

- Example: Can $\frac{3W+X^2+YZ}{Y}$ be simplified to $3W + X^2 + Z$?

    - Answer: No. This action would involve "partial canceling" which is not correct.

A complex fraction is a fraction that has other fractions in it. You should be able to simplify a complex fraction so that there are no more than two faction "levels" in it. One approach is to multiply the numerator and denominator by the common denominator of all the "little" fractions.

- Example: Write $\frac{1+\frac{2}{X}}{1-\frac{4}{X^2}}$ First, multiply both numerator and denominator of the complex fraction by $X^2$ to get $\frac{X^2+2X}{X^2-4}$.

**Note: Multiplying by $\frac{X^2}{X^2}$ is a convenient way of multiplying by 1 which is a permissible operation when working with expressions.**

Next, factor the numerator and denominator so that you have $\frac{X(X+2)}{(X-2)(X+2)}$. Next, cancel $(X+2)$ which is a factor in both the numerator and denominator. Finally, the initial expression, is simplest form, is $\frac{X}{X+2}$.

- Does $\frac{1}{X+Y} = \frac{1}{X} + \frac{1}{Y}$?

    - Answer: No. Try substituting simple numbers for X and Y to verify that the two sides of the above equation would not equal each other.

- Does $\frac{X-Z}{X} = 1 - \frac{Z}{X}$
  - Answer: Yes, because division distributes over subtraction. This can be verified by substituting simple numbers for X and Z.

# Special products and factoring

You should have memorized some special forms and be able to write the results from memory without having to multiply out all the pieces and then simplifying.

- $a^2 - b^2 = (a+b)(a-b)$: This pattern is called the "difference of two squares." You may be called upon to factor repeatedly using this pattern whenever it's possible.
  - Example:

$$
\begin{aligned}
X^4 - 16 \ &= \\
&= \ (X^2 - 4)(X^2 + 4) \ [\text{difference of two squares}] \\
&= \ (X - 2)(X + 2)(X^2 + 4)
\end{aligned}
$$

- Squaring a binomial

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

Example: Doing only mental calculations, can you write the results of squaring $(3X - 2Y)$?

- Answer: Using the above pattern, the result is: $9X^2 - 12XY + 4Y^2$.

# First derivatives

A function, $y = f(X)$, is a rule that specifies the relationship between two variables, where Y is the dependent variable and X is the independent variable. The set of values from which the independent variable is drawn is called the domain of the function, while the set of values assigned to the dependent variable is called the range. The function $Y = f(X)$ specifies a unique value in the range of Y for each value in the domain of X. In business applications the domain and the range are usually real-values as opposed to imaginary numbers. For example, variables such as prices, quantities, etc., are non-negative. Thus, the domain and range may be limited to non-negative real numbers.

## Common functions found in business

| Function | Description |
|---|---|
| $Y = a$ | constant function |
| $Y = aX$ | linear function |
| $Y = aX^2 + bX + C$ | quadratic function |
| $Y = a_0 + a_1X + a_2X^2 + a_3X^3 + ... + a_nX^n$ | polynomial function of degree $n$ |
| $Y = ae^x$ | exponential function |
| $Y = a\ ln(X)$ | logarithmic function |

## Differential Calculus

Differential calculus, despite its complex sounding name, is really nothing more than "slope finding." The slope of a function at a point $Y_0 = f(X_0)$ is the slope of the line that is tangent to the function $Y = f(X)$ at the point $(X_0, Y_0)$. If $\Delta Y$ and $\Delta X$ denote the changes in variables Y and X, respectively, then the approximate value of the slope of a function at a point $(X_0, Y_0)$ is given by:

$$\frac{\Delta Y}{\Delta X} = \frac{f(X_0 + \Delta X) - f(X_0)}{\Delta X}$$

This slope is approximate because a tangent line slope is defined at a single point $(X_0, Y_0)$, where X and Y are fixed and do not change, whereas the formula allows X and Y to vary away from point $(X_0, Y_0)$. The degree of discrepancy between a tangent line slope and the slope formula will depend on the size of $\Delta X$. As $\Delta X$ becomes smaller, the slope formula and the tangent line slope converge on the same value.

The derivative of a function at a point $Y_0 = f(X_0)$ is the rate of change, or slope of the tangent line, of $Y = f(X_0)$ at the point $(X_0, Y_0)$. The value of the derivative is given by the limit of the slope function, which is denoted either by $\frac{dY}{dX}$ or by $f'(X)$. Thus, we have:

$$\frac{dY}{dX} = \lim_{\Delta X \to 0} \frac{f(X_0 + \Delta X) - f(X_0)}{\Delta X}$$

If this limit exists (from both sides), the function is said to be a differentiable function.

## Selected rules of differentiation

| Derivative | Description |
|---|---|
| $Y = aX^0 \to \frac{dY}{dX} = 0$ | constant rule |
| $Y = aX^b \to \frac{dY}{dX} = baX^{b-1}$ if $b \neq 0$ | Power function rule |
| $Y = g(X) + h(X) \to \frac{dY}{dX} = g'(X) + h'(X)$ | Sum-difference rule |
| $Y = g(X) \times h(X) \to \frac{dY}{dX} = g'(X) \times h(X) + g(X) \times h'(X)$ | Product Rule |
| $Y = \frac{g(X)}{h(X)} \to \frac{dY}{dX} = \frac{g'(X) \times h(X) - g(X) \times h'(X)}{h(X)^2}$ | Quotient rule |

# Optimization

## Second and higher-order derivatives

One of the most widely used mathematical tools in economics and business is optimization analysis. For a single-variable function, $Y = f(X)$, the local maximum or minimum will be found where the first derivative is equal to zero. This is called the first-order condition. First-order conditions are necessary for a local extreme point, but they are not sufficient to indicate whether the extreme point is a minimum or a maximum. To check whether a zero value of a derivative is a maximum or a minimum, we need to check the second-order condition. The second-order condition is the sign of the second derivative of the original function.

- For local maximums, the two requirements are:

$$\frac{dY}{dX} = 0$$

$$\frac{d^2Y}{dX^2} < 0$$

- For local minimums, the two requirement are:

$$\frac{dY}{dX} = 0$$

$$\frac{d^2Y}{dX^2} > 0$$

## Example 1:

Suppose Fellini's Fabric Shop sells discount material. Its demand and cost functions are $Q = 40 - 2P$ and $TC = 0.5Q^2$ where $Q$ is expressed in linear yards and price is in dollars per linear yard. Determine the profit-maximizing quantity and price, and maximum profit.

- What quantity should Fellini's sell to maximize profit?

First, convert the demand function which is in standard form to inverse form by solving for P.

$$P = 20 - 0.5Q$$

Second, find total revenue by multiplying price by quantity.

$$
\begin{aligned}
TR &= P \times Q \\
TR &= (20 - 0.5Q)Q \\
TR &= 20Q - 0.5Q^2
\end{aligned}
$$

Third, find profit by subtracting total cost from total revenue.

$$
\begin{aligned}
\pi &= TR - TC \\
\pi &= 20Q - 0.5Q^2 - 0.5Q^2 \\
\pi &= 20Q - Q^2
\end{aligned}
$$

Fourth, take the first derivative of the profit function, set it equal to zero, and solve for Q to determine the quantity that maximizes (or minimizes) the function.

$$
\begin{aligned}
\pi &= 20Q - Q^2 \\
\frac{d\pi}{dQ} &= 20 - 2Q \\
0 &= 20 - 2Q \\
2Q &= 20 \\
Q &= 10
\end{aligned}
$$

Therefore, 10 is the quantity that maximizes, or minimizes, profit.

Fifth, find the value of the second derivative. If negative, then $\pi$ is maximum; if positive, $\pi$ in minimum.

$$
\begin{aligned}
\frac{d\pi}{dQ} &= 20 - 2Q \\
\frac{d^2\pi}{dQ^2} &= -2
\end{aligned}
$$

Since the second derivative is negative, $\pi$ is maximum at $Q = 10$. What price will Fellini's charge at the profit maximizing quantity? First, substitute the profit-maximizing quantity into the inverse form of the demand equation to determine the price that Fellini will charge at the profit-maximizing quantity.

$$\begin{aligned} P &= 20 - 0.5Q \\ P &= 20 - 0.5(10) \\ P &= 20 - 5 \\ P &= 15 \end{aligned}$$

Therefore, price at the profit-maximizing quantity is \$15.

What is Fellini's maximum profit for the specified time period?

First, substitute the profit-maximizing quantity into the profit function to determine maximum profit.

$$\begin{aligned} \pi &= 20Q - Q^2 \\ \pi &= 20(10) - (10)^2 \\ \pi &= 200 - 100 \\ \pi &= 100 \end{aligned}$$

Therefore, Fellini's maximum profit for the specified time period is \$100.

## Example 2:

Suppose Fellini's Fabric Shop, rather than maximizing profits, wants to maximize total revenue. Again, Fellini's demand and cost functions are $Q = 40 - 2P$ and $TC = 0.5Q^2$ where $Q$ is expressed in linear yards and price is in dollars per linear yard. Determine the total revenue-maximizing quantity, the price that must be charged to sell that quantity. Also, let's determine Fellini's profits when it maximizes total revenue to see how they compare to profits when Fellini's is attempting to maximize profits.

First, we are attempting to maximize total revenue, so we only need to begin with the total revenue function from the previous problem.

$$\begin{aligned} TR &= P \times Q \\ TR &= (20 - 0.5Q)Q \\ TR &= 20Q - 0.5Q^2 \end{aligned}$$

Second, find the first derivative, set it equal to 0, and solve for Q to determine the quantity that maximizes (or minimizes) the function.

$$\begin{aligned}
TR &= 20Q - 0.5Q^2 \\
\frac{dTR}{dQ} &= 20 - Q \\
0 &= 20 - Q \\
Q &= 20
\end{aligned}$$

Therefore, 20 is the quantity that maximizes (or minimizes) the function.

Third, let's determine the price that we must charge to be able to sell 20 units of output by substituting the total revenue-maximizing (or minimizing) quantity into the demand function.

$$\begin{aligned}
P &= 20 - 0.5Q \\
&= 20 - 0.5(20) \\
&= 20 - 10 \\
&= 10
\end{aligned}$$

Note that the price of $10 necessary to maximize (or minimize) total revenue is different from the price of $15 that was charged to maximize profit.

Fourth, substitute $Q = 20$ into the total revenue function to determine maximum (or minimum) total revenue.

$$\begin{aligned}
TR &= 20Q - 0.5Q^2 \\
&= 20(20 - 0.5(20)^2 \\
&= 400 - 0.5(400) \\
&= 400 - 200 \\
&= 200
\end{aligned}$$

Therefore, 200 is either the maximum or minimum value for total revenue.

Fifth, determine the value of the second derivative of the total revenue function. If negative, the function has a maximum value of 200; if positive, the function has a minimum value of 200.

$$\begin{aligned}
TR &= 20Q - 0.5Q^2 \\
\frac{dTR}{dQ} &= 20 - Q \\
\frac{d^2TR}{dQ^2} &= -1
\end{aligned}$$

Since the second derivative is negative, the function is maximum.

Sixth, let's determine profit at the total revenue-maximizing quantity to see how it compares to maximum profit. Obviously, since we previously determined the quantity that maximizes profit, and the quantity that maximizes total revenue is a different quantity, we expect the profit to be less.

$$\begin{aligned} \pi &= 20Q - Q^2 \\ &= 20(20) - (20)^2 \\ &= 400 - 400 \\ &= 0 \end{aligned}$$

Therefore, profit is 0 at the total revenue-maximizing quantity whereas profit was 100 at the profit-maximizing quantity.

*Now, let's summarize these two examples. First, if the firm attempts to maximize profits, it needs to sell 10 units, and can do so at a price of \$15, which will then generate maximum profits of \$100. Second, if the firm attempts to maximize total revenue, it needs to sell 20 units, and can do so by charging a price of \$10, but the firm will earn \$0 profits.*

## Probability and Integral Calculus

Probability theory is fundamental part of modern statistics. One of the most important notions comes from integral calculus. Integral calculus serves us to compute area under the probability functions, $f(x)$. This area under the probability function constitutes the probability of an event.
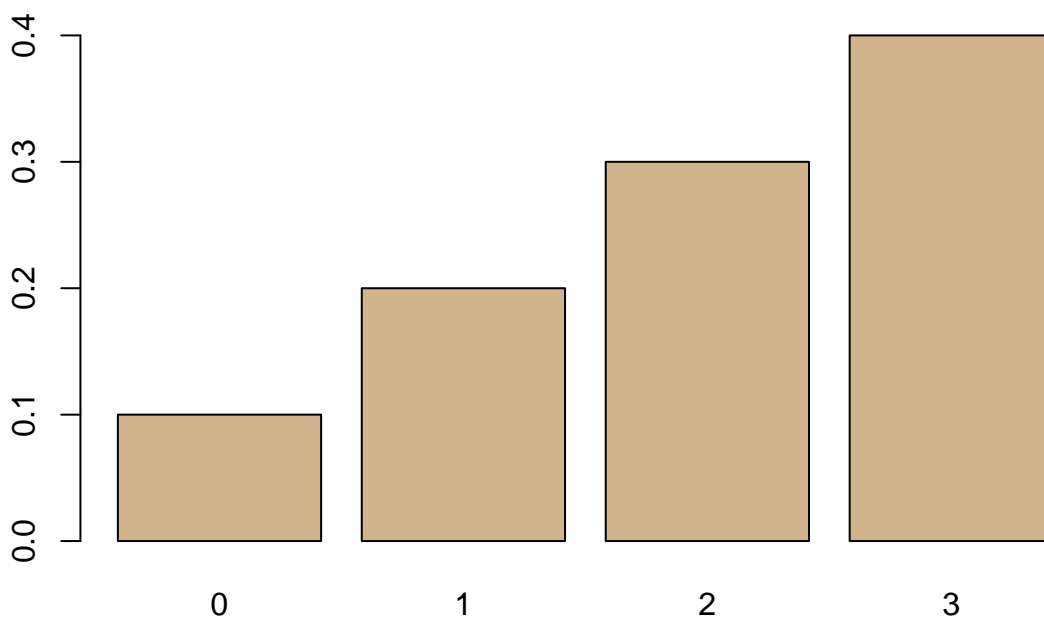
- If the area under the curve is discrete, we calculate the are under the curve by adding...

$P(X = 0) = 0.10$, $P(X = 1) = 0.20$, and $P(X = 2) = 0.30$

then

$P(0 \leq X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \sum_{i=0}^{2} P(X = i)$

- If the area under the curve is continuous, we calculate the are under the curve by integrating...

## Example

- A soft drink machine outputs a mean of 24 ounces per cup. The machine's output is known to be distributed Gaussian with a standard deviation of two ounces. What is the probability of putting less than 27 ounces in a cup?

**Soft drink machine output**
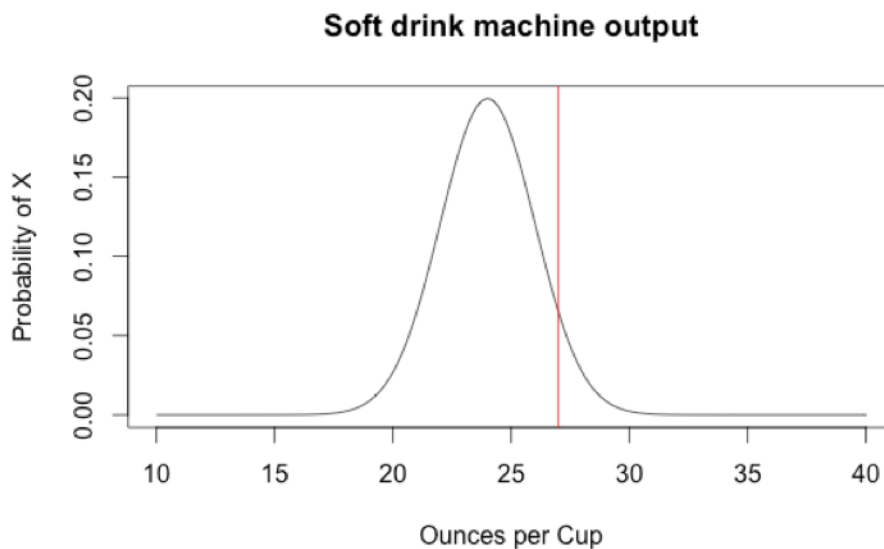


$$P(X < 27 \text{ ounces} \mid \mu = 24, \sigma = 2) = \int_{-\infty}^{27} P(x)dx$$

where,

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

```
pnorm(27,mean=24, sd=2)
```

```
[1] 0.9331928
```

# Software

- **R Laguage:** The software used for the computer-calculated examples shown in the previous text is R. R is an open source language for statistical programming: https://www.r-project.org/.

- **Maxima & wxMaxima:** If you need a symbolic mathematical package, there is an open source package called *Maxima*. Maxima is a system for the manipulation of symbolic and numerical expressions, including differentiation, integration, Taylor series, Laplace transforms, ordinary differential equations, systems of linear equations, polynomials, sets, lists, vectors, matrices and tensors: http://maxima.sourceforge.net and http://andrejv.github.io/wxmaxima/.